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The realization of a generalized spin measurement

D T Pegg

Centre for Quantum Computer Technology, School of Biomolecular and Physical Sciences, Griffith University, Nathan, Brisbane 4111, Australia

E-mail: d.pegg@griffith.edu.au

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Abstract
A generalized spin measurement, as opposed to a projective measurement, can produce interesting effects such as the weak Zeno effect (Cresser *et al* 2006 *Opt. Commun.* **264** 352). In the weak limit a generalized spin measurement can also lead to surprising weak values (Aharonov *et al* 1988 *Phys. Rev. Lett.* **60** 1351). In this paper we show explicitly how a generalized spin measurement can be realized physically by entangling the spin to be measured with a pointer spin. Adjustment of the interaction time allows any amount of generalization to be achieved, ranging from a projective measurement to the limit of a very weak measurement.

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1. Introduction

The probability of an outcome i of a quantum measurement on a system is the trace $\mathrm{Tr}(\hat{\rho}\hat{\pi}_i)$, where $\hat{\rho}$ is the density operator of the system. For a von Neumann projective measurement $\hat{\pi}_i$ is a projector. For more general quantum measurements $\hat{\pi}_i$ is an element of a probability operator measure (POM) [1]. The subsequent change of state of the measured system is determined by an associated Kraus operator $\hat{A}_i$ such that [2]

$$\hat{\pi}_i = \hat{A}_i^\dagger \hat{A}_i \tag{1}$$

with the state after the measurement being given by

$$\frac{\hat{A}_i \hat{\rho} \hat{A}_i^\dagger}{\mathrm{Tr}(\hat{\rho}\hat{\pi}_i)}. \tag{2}$$

The simplest generalization of a projective measurement on a non-trivial quantum system can be represented by the POM elements

$$\hat{\pi}_1 = (1-p)|1\rangle\langle 1| + p|2\rangle\langle 2| \tag{3}$$

$$\hat{\pi}_2 = (1 - p)|2\rangle\langle 2| + p|1\rangle\langle 1|, \tag{4}$$

where $|1\rangle$ and $|2\rangle$ are the states of a two-state, or qubit, system. The sum of the two POM elements is the unit operator as required [1]. If the aim of the measurement is to determine the state, then the value of p can be regarded as the probability that there is an error in the measurement. For example p can be regarded as the probability that a measurement performed on the system in state $|2\rangle$ will indicate that the system is in state $|1\rangle$. The two limiting cases $p = 0$ and $p = 1/2$ correspond to a completely accurate, projective, measurement and a completely indeterminate measurement respectively. In the latter case the POM elements are proportional to the unit operator and the measurement tells us nothing about the state of the system. When p is slightly less than $1/2$, the measurement is weak. In this case only a small amount of information is gained from the measurement. Weak measurements can be used in conjunction with postselection to find weak values, which can have the interesting property of being well outside the normal range of measurement results, for example spins of 100 for a spin-half particle [3].

The Kraus operators associated with the POM elements $\hat{\pi}_i$ are not completely determined by these POM elements but the simplest case, in which the Kraus operators are Hermitian, is easily found as in the next section. With these Kraus operators a measurement master equation can be derived for the evolution of a driven two-level system, such as a spin-half particle, while subjected to a sequence of measurements [4]. Analysis of this evolution revealed a range of interesting trajectories from Rabi precession through to quantum Zeno type suppressed evolution [5] as the type of measurement was varied from very weak to strong frequent projective measurements [4]. For accurate projective measurements at frequent intervals a random telegraph results with the population inversion remaining at $+1$ or -1 nearly all of the time with measurement induced sudden quantum jumps between the two values. In between the two extremes of Rabi precession and random telegraph is an unusual and interesting phenomenon referred to in [4] as the weak Zeno effect. This shows a type of random telegraph behaviour, but with many aborted jumps between successful jumps. The weak Zeno effect is also apparent in the quantum trajectories calculated in [6].

It is not difficult to show (for example, see [7]) that any general measurement can be realized mathematically by means of an ancillary system, a unitary transformation and a projective measurement. However, whether or nor such a measurement and the associated Kraus operators can be physically realized is another question. In this paper, we show how the general POM elements (3) and (4) and the associated Kraus operators (5) and (6) that give rise to the weak Zeno effect can be realized physically from a projective spin-half measurement. Such a projective measurement on a single spin is, of course, a challenge in itself, but progress is being made [8, 9] with more progress to be expected because of the importance of such a measurement for quantum computing.

2. Kraus operators

In the derivation [4] of the master equation describing the evolution of a driven two-level quantum system, or qubit, subjected to a sequence of imperfect generalized measurements of the energy level, it was assumed that these measurements could be represented by the POM elements (3) and (4) with $|2\rangle$ and $|1\rangle$ being the upper and lower energy states of the qubit. To find the measurement master equation it was necessary to know the state of the qubit after each measurement. The Kraus operators that determine this obey expression (1) but this property alone is insufficient to determine them uniquely. In [4] the forms

$$\hat{A}_1 = \sqrt{(1 - p)}|1\rangle\langle 1| + \sqrt{p}|2\rangle\langle 2| = \hat{A}_1^{\dagger} \tag{5}$$

$$\hat{A}_2 = \sqrt{(1-p)}|2\rangle\langle 2| + \sqrt{p}|1\rangle\langle 1| = \hat{A}_2^{\dagger} \tag{6}$$

were chosen, from the infinite number of possible Kraus operators, on the basis of their simplicity. In the limit of a very weak measurement, $p = 1/2$, these operators are proportional to the unit operator so their action does not affect the state of the system. Kraus operators of a similar form were also employed in [10].

The question arises as to what sort of measurement scheme can possibly yield the POM elements (3) and (4) and the Kraus operators (5) and (6). A quantum optical way in which similar operators might be realized by means of cavity quantum electrodynamics was suggested in [11]. The system we wish to study here is where the qubit is represented by a spin-half system in a magnetic field in the z-direction. In this case the POM elements for a direct accurate measurement of the z-component of spin $\hat{I}_z$ would be $|1\rangle\langle 1|$ and $|2\rangle\langle 2|$. A property of (5) and (6) is that if the system is initially in state $|1\rangle$, say, then even an inaccurate measurement, that is with $p \neq 0$, does not affect the state of the system. This excludes modelling an inaccurate measurement by a mechanism involving measuring the z'-component of $\hat{I}_z$, where z' is not accurately aligned with z. In this paper we develop a physical model that realizes the required POM elements and Kraus operators by a more indirect measurement. Here we couple for a short time another spin-half particle representing a 'pointer' spin to the spin-half particle representing the qubit to be measured. After this, the pointer spin is measured by a projective measurement.

3. Pointer spin

As mentioned previously, the problem of measuring the orientation of a single spin by a projective measurement, while very important for the development of a practical quantum computer, is not the problem we address here. Here we assume that this can be done and the problem we address is that of performing a controllable generalized measurement of spin described by the Kraus operators (5) and (6). We study this in terms of a model that couples an ancillary spin-half particle to the qubit to be measured. The ancillary spin is then subsequently subjected to a projective measurement. In this way, the ancillary spin acts as a pointer. For the interaction with the pointer not to affect the spin under examination if it is in the state $|1\rangle$ or $|2\rangle$, these states must be eigenstates of the interaction Hamiltonian.

Consider two spin-half nuclei, such as a proton and a carbon-13 nucleus, with different gyromagnetic ratios. The two spins will have a scalar coupling with interaction energy $2\pi J\hat{\mathbf{I}}\cdot\hat{\mathbf{S}}$, where $\hat{\mathbf{I}}$ and $\hat{\mathbf{S}}$ are the spin vectors for the proton and carbon nucleus respectively and J depends on the distance between the spins [12]. A strong magnetic field is applied in the z-direction and a radio-frequency field is resonant with the proton. We wish to use the carbon-13 nucleus as the pointer spin. In a strong field the precessional frequencies of the two spins are very different so only the contribution to $\hat{\mathbf{I}}\cdot\hat{\mathbf{S}}$ from $\hat{I}_z\hat{S}_z$ is static with the other terms $\hat{I}_x\hat{S}_x$ and $\hat{I}_y\hat{S}_y$ being averaged to zero very quickly. The term $\hat{I}_z\hat{S}_z$ can also be averaged quickly to zero by dressing the carbon nucleus with a strong, reasonably broad-band noise-modulated radiofrequency field centred on the carbon resonance frequency but well removed from the proton resonance frequency. The resultant rapid carbon spin flipping averages the value of $\hat{I}_z\hat{S}_z$ to zero, effectively switching off the coupling entirely. This decoupling procedure works well and is a standard technique in nuclear magnetic resonance [13, 14].

We shall work in the doubly rotating reference frame [15] by transforming from the laboratory frame with the unitary transformation $\exp(-\mathrm{i}\hat{I}_z\omega_p t)\exp(-\mathrm{i}\hat{S}_z\omega_c t)$, where ω_p and ω_c are the resonance frequencies of the proton and carbon spins. In order to save using subscripts on the kets, we shall label the proton eigenstates of $\hat{I}_z$ with numbers, that is $|1\rangle$ and

$|2\rangle$, and use letters to label the pointer states in the doubly rotating frame. For example $|\pm y\rangle$ will represent the state of the pointer spin in the $\pm y$ direction of the doubly-rotating frame. In this frame with the coupling switched off, that is, with the decoupling field switched on, the proton undergoes Rabi precession between states $|1\rangle$ and $|2\rangle$. To approximate an instantaneous measurement, we let the Rabi precession be sufficiently slower than the motion induced by the coupling for us to ignore the Rabi precession for the brief time that the coupling is switched on and then off again. We assume that we can prepare the initial state of the carbon (pointer) spin, which will be $|-y\rangle$, in a very short time by first projectively measuring the z-component of this spin and then applying a hard $\pi/2$ pulse about the axis appropriate to the result obtained. The procedure is to switch on the coupling, quickly put the pointer spin in the initial state $|-y\rangle$, allow time for the pointer states to become entangled with the proton states, measure $\hat{S}_x$ of the pointer and then switch the coupling off again.

3.1. Projective measurement

Before examining how to make a generalized measurement of the proton spin, we consider the simpler case where we wish to use a projective measurement of the carbon pointer spin to yield a projective measurement of the proton spin. When the coupling is switched on, the Hamiltonian in the doubly-rotating frame is $2\pi J\hat{I}_z\hat{S}_z$ where, as mentioned above, we are allowing the Rabi precession to be sufficiently slow to be ignored for the period τ that the coupling is switched on. From the time-dependent Schrödinger equation, the unitary time evolution operator for the interaction period τ is given by

$$\exp(-\mathrm{i}2\pi J\hat{I}_z\hat{S}_z\tau) = \cos(\pi J\tau/2) - \mathrm{i}4\hat{I}_z\hat{S}_z\sin(\pi J\tau/2). \tag{7}$$

This result can obtained by expanding the exponential series and remembering that $\hat{I}_z^2 = \hat{J}_z^2 = 1/4$ for spin-half particles [15]. To make a projective measurement of the proton spin we choose the interaction time to be $\tau = (2J)^{-1}$.

We let the initial state of the pointer spin be $|-y\rangle$ and the proton be in a general superposition state $\alpha|2\rangle + \beta|1\rangle$. After the interaction time $(2J)^{-1}$ the combined state will have evolved to $\alpha|A\rangle + \beta|B\rangle$, where

$$|A\rangle = 2^{-1/2}[1 - \mathrm{i}4\hat{I}_z\hat{S}_z]|2\rangle|-y\rangle \tag{8}$$

$$|B\rangle = 2^{-1/2}[1 - \mathrm{i}4\hat{I}_z\hat{S}_z]|1\rangle|-y\rangle. \tag{9}$$

Using the relations $|\pm y\rangle = 2^{-1/2}(|z\rangle \pm -\mathrm{i}|-z\rangle)$ and $|\pm x\rangle = 2^{-1/2}(|z\rangle \pm |-z\rangle)$ [16], we obtain

$$|A\rangle = \exp(-\mathrm{i}\pi/4)|2\rangle 2^{-1/2}(|z\rangle + |-z\rangle) = \exp(-\mathrm{i}\pi/4)|2\rangle|x\rangle \tag{10}$$

$$|B\rangle = \exp(\mathrm{i}\pi/4)|1\rangle 2^{-1/2}(|z\rangle - |-z\rangle) = \exp(\mathrm{i}\pi/4)|1\rangle|-x\rangle. \tag{11}$$

The final combined state is thus the entangled state

$$|F\rangle = \alpha\exp(-\mathrm{i}\pi/4)|2\rangle|x\rangle + \beta\exp(\mathrm{i}\pi/4)|1\rangle|-x\rangle. \tag{12}$$

We note that this result is in accord with the result of an ideal quantum measurement procedure described by Wigner [17]. If, for example, the proton is initially in the state $|2\rangle$, that is if $\alpha = 0$, then the interaction does not change this state but does change the state of the pointer in a way dependent on the proton state. The next step is to make a projective measurement of $\hat{S}_x$, the x-component of the pointer spin. As stated earlier, we shall assume that this can be done and so will not discuss this further here except to say that if it is easier to measure $\hat{S}_z$, the component in the direction of the strong magnetic field, then we could apply a suitable $\pi/2$ pulse to the pointer spin immediately preceding the measurement. A measurement of $\hat{S}_x$ has two possible results, corresponding to measurement events with POM elements $|-x\rangle\langle -x|$ and $|x\rangle\langle x|$. As the proton is not measured, the associated POM contains just one element $\hat{1}_\mathrm{p}$, the unit operator acting on the proton state space. The probability for the first of the two possible pointer spin measurement events is thus

$$\mathrm{Tr}(|F\rangle\langle F|\hat{1}_\mathrm{p}|-x\rangle\langle -x|) = \mathrm{Tr}_\mathrm{p}(\langle -x|F\rangle\langle F|-x\rangle), \tag{13}$$

where Tr is the trace over both the proton and pointer states and Tr_p is the trace over the proton states. From (12) we then find that the probability of the first pointer spin measurement event is

$$\mathrm{Tr}_\mathrm{p}(|\beta|^2|1\rangle\langle 1)| = \mathrm{Tr}_\mathrm{p}[(\alpha|2\rangle + \beta|1\rangle)(\alpha^*\langle 2| + \beta^*\langle 1|)|1\rangle\langle 1|], \tag{14}$$

so the effective POM element $\hat{\pi}_1$ for this event is $|1\rangle\langle 1|$, which is expression (3) with $p = 0$. Likewise we find that the effective POM element $\hat{\pi}_2$ for the second measurement event is $|2\rangle\langle 2|$. The resulting probabilities for these measurement events are precisely the same as we would obtain if we were to measure the z-component of spin of the proton in state $\alpha|2\rangle + \beta|1\rangle$ directly.

Depending on whether the measurement projects the state $|F\rangle$ onto the pointer state $|-x\rangle$ or $|+x\rangle$, the unnormalized state of the proton after the measurement will be either

$$\langle -x|F\rangle = \beta\exp(\mathrm{i}\pi/4)|1\rangle \tag{15}$$

or

$$\langle x|F\rangle = \alpha\exp(-\mathrm{i}\pi/4)|2\rangle. \tag{16}$$

If we wished to, we could eliminate the phase factors $\exp(\pm\mathrm{i}\pi/4)$ by applying a $\pi/2$ pulse about the x-axis to the pointer spin before measurement. However, there is no need to do this as these factors cancel in calculating the post-measurement density operator as in [4]. Then appropriate Kraus operators for use with the density operator are

$$\hat{A}_1 = |1\rangle\langle 1| = \hat{A}_1^\dagger \tag{17}$$

$$\hat{A}_2 = |2\rangle\langle 2| = \hat{A}_2^\dagger, \tag{18}$$

which can easily be checked. For example, the action of $\hat{A}_1$ on the state $\alpha|2\rangle + \beta|1\rangle$ produces the state (15) without the phase factor. These Kraus operators are in agreement with (5) and (6) with $p = 0$.

3.2. Generalized measurement

To model a generalized measurement of the proton spin, we let the interaction time τ deviate from $(2J)^{-1}$ so that now

$$\pi J\tau/2 = \pi/4 - \epsilon, \tag{19}$$

where ϵ can be positive or negative. The time evolution operator is now

$$\exp(-\mathrm{i}\pi\hat{I}_z\hat{S}_z)\exp(\mathrm{i}4\hat{I}_z\hat{S}_z\epsilon) = (\cos\epsilon + \mathrm{i}4\hat{I}_z\hat{S}_z\sin\epsilon)\times\exp(-\mathrm{i}\pi\hat{I}_z\hat{S}_z). \tag{20}$$

so the state $|A\rangle$ given by (10) is now replaced by

$$|A'\rangle = (\cos\epsilon + \mathrm{i}4\hat{I}_z\hat{S}_z\sin\epsilon)|A\rangle. \tag{21}$$

Using $2\hat{S}_z|\pm x\rangle = |\mp x\rangle$ gives, from (10),

$$|A'\rangle = \exp(-\mathrm{i}\pi/4)|2\rangle(\cos\epsilon|x\rangle + \mathrm{i}\sin\epsilon|-x\rangle). \tag{22}$$

Similarly we find

$$|B'\rangle = \exp(\mathrm{i}\pi/4)|1\rangle(\cos\epsilon|-x\rangle - \mathrm{i}\sin\epsilon|x\rangle) \tag{23}$$

with the final entangled state $|F'\rangle$ being $\alpha|A'\rangle + \beta|B'\rangle$. A measurement of $\hat{S}_x$ then projects the final state onto $|\mp x\rangle$, leaving the proton either in the state

$$\begin{aligned}\langle -x|F'\rangle &= \exp(\mathrm{i}\pi/4)\beta\cos\epsilon|1\rangle + \mathrm{i}\exp(-\mathrm{i}\pi/4)\alpha\sin\epsilon|2\rangle \\ &= \exp(\mathrm{i}\pi/4)(\beta\cos\epsilon|1\rangle + \alpha\sin\epsilon|2\rangle)\end{aligned} \tag{24}$$

or in the state

$$\langle x|F'\rangle = \exp(-\mathrm{i}\pi/4)(\alpha\cos\epsilon|2\rangle + \beta\sin\epsilon|1\rangle). \tag{25}$$

The probability with which the first measurement event occurs is given by (13) with $|F\rangle$ replaced by $|F'\rangle$, which we can write as

$$\mathrm{Tr}_{\mathrm{p}}[(\alpha|2\rangle + \beta|1\rangle)(\alpha^*\langle 2| + \beta^*\langle 1|)\hat{\pi}_1], \tag{26}$$

where

$$\hat{\pi}_1 = \cos^2\epsilon|1\rangle\langle 1| + \sin^2\epsilon|2\rangle\langle 2|. \tag{27}$$

Similarly we find that the effective POM element for the second measurement event is

$$\hat{\pi}_2 = \sin^2\epsilon|1\rangle\langle 1| + \cos^2\epsilon|2\rangle\langle 2|. \tag{28}$$

The probability for the first event is $|\alpha|^2\sin^2\epsilon + |\beta|^2\cos^2\epsilon$ and for the second event is $|\alpha|^2\cos^2\epsilon + |\beta|^2\sin^2\epsilon$. Clearly these POM elements are of the required form (3) and (4) with $p = \sin^2\epsilon$.

In determining the Kraus operators we can ignore the phase factors $\exp(\pm\mathrm{i}\pi/4)$ as before and obtain

$$\hat{A}_1 = \cos\epsilon|1\rangle\langle 1| + \sin\epsilon|2\rangle\langle 2| \tag{29}$$

$$\hat{A}_2 = \cos\epsilon|2\rangle\langle 2| + \sin\epsilon|1\rangle\langle 1|. \tag{30}$$

Again, these can easily be checked, for example the action of $\hat{A}_1$ in (29) on the state $\alpha|2\rangle+\beta|1\rangle$ produces the unnormalized state (24) without the phase factor. These Kraus operators are in agreement with (5) and (6) with $p = \sin^2\epsilon$. It is clear that if $\epsilon = \pi/4$ then the Kraus operators are proportional to the unit operator and so the measurement does not affect the state. From (19) this is so when the coupling time τ is zero so no entanglement has built up between the proton and the pointer spin. We have the same situation when $\epsilon = -3\pi/4$. This is when the coupling time is such that the entanglement first builds up and then reduces to zero. Small deviations of ϵ from $\pi/4$ or $-3\pi/4$ result in weak measurements.

4. Conclusion

We have shown that the POM elements and Kraus operators that give interesting effects such as the weak Zeno effect described in [4] or weak values [3] can be achieved, at least in principle, by coupling an ancillary spin to the main spin, that is, the spin under examination. This ancillary spin acts as a pointer indicating the state of the main spin. By adjusting the coupling time appropriately, the measurement can be varied from a projective, or von Neumann, measurement to a very weak measurement that barely affects the main spin.

As viewed from the doubly rotating reference frame, the measurement process has a straightforward interpretation. The pointer spin initially points along the negative y-axis of this frame. From (10) and (11), after a time $\tau = (2J)^{-1}$, the pointer rotates to point in the positive x-direction if the main spin is in the 'up' state $|2\rangle$ or in the negative x-direction if the main spin is in the down state $|1\rangle$. If the main spin is in a superposition state, the states of the two spins become entangled and a particular outcome of the measurement leaves the main spin in the associated state. The unimportant phase factors in (10) and (11) have a simple interpretation in terms of the rotating pointer. If the coupling is allowed to remain on for four times the above period, the pointer will have rotated through 2π back to the negative y-axis and will have returned to its original state except for a phase factor of $\exp(\pm i\pi) = -1$. This is due to the spinor nature of the pointer. If we had used a spin-1 particle, for example, as the pointer, the phase factor would be different.

A coupling time of $\tau = (2J)^{-1}$ as above leads to a projective, or von Neumann, type of measurement. To obtain a controllable generalized measurement, we simply reduce this period so the main spin and pointer states are not so strongly entangled. When the entanglement is very small a weak measurement results. It is worth noting that a weak measurement does not require a weak coupling. Indeed the stronger the coupling the better for the purposes used in [4], because this allows a more rapid measurement to be made. Reduction of the coupling time leads to measurements described by (3), (4), (5) and (6) as required in [4] where p is determined by the chosen coupling time. Overall, the method presented here is a reasonably straightforward way of physically realizing, at least in principle, a controllable generalized measurement.

Finally, in comparing the procedure with Wigner's ideal measurement procedure, we did not mention that Wigner's pointer was macroscopic [17]. In our case the pointer being measured is just as microscopic as the system being measured. Our procedure can, however, be readily extended. Consider, for example, a spin-half silicon-29 nucleus in a $Si(CH_3)_4$ molecule. The qubit in this case is surrounded by 12 equivalent protons which together can function as the pointer. It is not difficult to show that this pointer moves in a similar way to the single-spin pointer considered in this paper. While a 12-proton pointer is hardly macroscopic, or even mesoscopic, its measurement would be easier than that of a single spin.

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